Existence and Regularity for Dirichlet Problems

Bokuan (Jerry) Li

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Contents

Notations

Differential geometry is the study of properties that are invariant under change of notation. Here are the ones in this document that may require clarification:

$(\overline{\Omega}, g)$	d-dimensional Riemannian manifold
g^{mn}	g in local coordinates
$\Omega, \partial\Omega$	Interior and boundary of $\overline{\Omega}$
$\langle \cdot, \cdot \rangle_g$	Inner product given by g
$L^2(\Omega; T\Omega)$	L^2 space of vector fields
$\langle \cdot, \cdot \rangle_{L^2(\Omega;T\Omega)}$	Inner product on $L^2(\Omega; T\Omega)$ given by g
$\Delta = \text{div grad}$	Laplace-Beltrami operator
$\mathfrak{X}(\Omega)$	Space of smooth vector fields
$(U, \varphi), \widehat{U}, \widehat{f}$	Coordinate chart, $\widehat{U} = \varphi(U)$, $\widehat{f} = f \circ \varphi^{-1}$
$U \subset\subset V$	$\overline{U}$ is a compact subset of V
$H^s(\Omega)$	Sobolev space of order s on Ω
$H^s_{\text{loc}}(\Omega)$	Localised Sobolev space of order s on Ω
$H^s_0(\Omega)$	Closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$
df	Differential of f
D_j	j-th partial derivative in local coordinates
D_j^h	j-th difference quotient of size h in local coordinates
D^α	α-th partial derivative in local coordinates
$\mathcal{D}(\Omega) = C_c^\infty(\Omega)$	Test functions

Whenever an estimate $||u||_E \leq C_{a,b,c,\cdots}(||u||_F + ||u||_G + \cdots)$ is given, the constant depends only on $a, b, c, \cdots$, which may change throughout a proof.

1 The Energy Estimate

Let $(\overline{\Omega}, g)$ be a smooth, Riemannian manifold, $\Delta = \operatorname{div} \operatorname{grad}$ be the Laplace-Beltrami operator, $X \in \mathfrak{X}(\overline{\Omega})$ be a smooth vector field, $c \in C^\infty(\overline{\Omega})$ be a smooth function, and L be the differential operator defined by

$$Lu = -\Delta u + Xu + cu$$

then there exists a map

$$C^\infty(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{R} \quad (u, \phi) \mapsto \langle u, \phi \rangle_L := \langle Lu, \phi \rangle_{L^2(\Omega)}$$

known as the **sesquilinear form associated with** L. By Green's formulas,

$$\langle Lu, \phi \rangle_{L^2(\Omega)} = \langle du, d\phi \rangle_{L^2(\Omega;T\Omega)} + \langle Xu, \phi \rangle_{L^2(\Omega)} + \langle cu, \phi \rangle_{L^2(\Omega)} \tag{1}$$

To evaluate $\langle u, \phi \rangle_L$, it is sufficient to differentiate both terms once. This yields an estimate

$$\begin{aligned} |\langle u, \phi \rangle_L| &\leq C_X \, ||du||_{L^2(\Omega;T\Omega)} \, ||\phi||_{H^1(\Omega)} + C_c \, ||u||_{L^2(\Omega)} \, ||\phi||_{L^2(\Omega)} \\ &\leq C_L \, ||u||_{H^1(\Omega)} \, ||u||_{H^1(\Omega)} \end{aligned}$$

so $\langle \cdot, \cdot \rangle_L$ is a sesquilinear form on $H^1(\Omega) \times H^1_0(\Omega)$, which defines an operator

$$L : H^1(\Omega) \to H^{-1}(\Omega) \quad u \mapsto \langle \cdot, u \rangle_L$$

If $L = -\Delta$ is simply the Laplacian, then for any $u \in H^1_0(\Omega)$, $\langle u, u \rangle_L = ||du||^2_{L^2(\Omega;T\Omega)}$ is the Dirichlet energy. In addition, if Pointcaré's inequality holds, then $||u||^2_{H^1(\Omega)} \leq C_{\Omega,L} \langle u, u \rangle_L$. In the more general case, $\langle u, u \rangle_L$ contains the Dirichlet energy, but also a number of lower order terms, which can be offset to produce a similar estimate:

Lemma 1.1 (Energy Estimate, [3, Theorem 5.1.3], [2, Theorem 6.2.2])**.** Let $u \in H^1_0(\Omega)$, then

$$||u||^2_{H^1(\Omega)} \leq C_\Omega(\operatorname{Re}(\langle u, u \rangle_L) + (C_L + C'_\Omega) \, ||u||^2_{L^2(\Omega)})$$

where

1. If $X = 0$ and $c \geq 0$[1], then $C_L = 0$.
2. If every connected component of Ω has non-empty boundary, then $C'_\Omega = 0$.

Proof. By Cauchy's inequality with $\varepsilon = 1/(2C_X)$,

$$\begin{aligned} \langle u, u \rangle_L &= ||du||^2_{L^2(\Omega;T\Omega)} + \langle Xu, u \rangle_{L^2(\Omega)} + \langle cu, u \rangle_{L^2(\Omega)} \\ \operatorname{Re}(\langle u, u \rangle_L) &\geq ||du||^2_{L^2(\Omega;T\Omega)} - C_X \, ||du||_{L^2(\Omega;T\Omega)} \, ||u||_{L^2(\Omega)} - C_c \, ||u||^2_{L^2(\Omega)} \\ &\geq \frac{1}{2} \, ||du||^2_{L^2(\Omega;T\Omega)} - C_X \, ||u||^2_{L^2(\Omega)} - C_c \, ||u||_{L^2(\Omega)} \end{aligned}$$

[1]These conditions may be relaxed to a bound on $||X||_u$ and a lower bound on c if Pointcaré's inequality holds.

so

$$||du||^2_{L^2(\Omega;T\Omega)} \le 2(\langle u,u\rangle_L + C_L\,||u||^2_{L^2(\Omega)})$$

where C_L can be taken to be 0 if $X = 0$ and $c \ge 0$. Therefore

$$\begin{aligned}||u||^2_{H^1(\Omega)} &\le C_\Omega(||u||^2_{L^2(\Omega)} + ||du||^2_{L^2(\Omega;T\Omega)})\\ &\le C_\Omega(\mathrm{Re}\,(\langle u,u\rangle_L) + (C_L + C'_\Omega)\,||u||^2_{L^2(\Omega)})\end{aligned}$$

If every connected component of Ω has non-empty boundary, then by Pointcaré's inequality,

$$||u||^2_{H^1(\Omega)} \le C_\Omega\,||du||^2_{L^2(\Omega;T\Omega)} \le C_\Omega(\mathrm{Re}\,(\langle u,u\rangle_L) + C_L\,||u||^2_{L^2(\Omega)})$$

and C'_Ω can be taken to be 0. □

2 Regularity of Solutions

The estimate in Lemma 1.1 suggests that the regularity of u can be inferred from the regularity of Lu. More specifically,

Lemma 2.1 ([3, Theorem 5.1.3])**.** Let $u \in H^1_0(\Omega)$, then

$$||u||^2_{H^1(\Omega)} \le C_{\Omega,L}(||Lu||^2_{H^{-1}(\Omega)} + ||u||^2_{L^2(\Omega)})$$

Proof. By Lemma 1.1,

$$||u||^2_{H^1(\Omega)} \le C_\Omega(\mathrm{Re}\,(\langle u,u\rangle_L) + ||u||^2_{L^2(\Omega)})$$

Using Cauchy's inequality with $\varepsilon = 1/2(C_\Omega)$,

$$\begin{aligned}||u||^2_{H^1(\Omega)} &\le C_\Omega(||Lu||_{H^{-1}(\Omega)}\,||u||_{H^1(\Omega)} + ||u||^2_{L^2(\Omega)})\\ &\le \frac{1}{2}\,||u||^2_{H^1(\Omega)} + C'_\Omega(||Lu||^2_{H^{-1}(\Omega)} + ||u||^2_{L^2(\Omega)})\\ ||u||^2_{H^1(\Omega)} &\le 2C'_\Omega(||Lu||^2_{H^{-1}(\Omega)} + ||u||^2_{L^2(\Omega)})\end{aligned}$$

□

Assume for a moment that $\Omega = \mathbb{R}^d$, L has constant coefficients, and $Lu \in H^1(\mathbb{R}^d)$. For any $1 \le j \le d$, the derivative $D_j u$ can be computed using difference quotients. Thus for $h \ne 0$ sufficiently small, by applying Lemma 2.1 to $D^h_j u$,

$$||D^h_j u||_{H^1(\mathbb{R}^d)} \le C(||L(D^h_j u)||_{H^{-1}} + ||D^h_j u||_{L^2(\mathbb{R}^d)}) \le C'(||Lu||_{L^2(\mathbb{R}^d)} + ||u||_{H^1(\mathbb{R}^d)})$$

since L and D^h_j commute. Thus $||u||_{H^2(\mathbb{R}^d)} \le C(||D^h_j(Lu)||_{H^{-1}} + ||u||_{H^1(\mathbb{R}^d)})$. When the coefficients are not constant, the commutator can be bounded as follows.

Lemma 2.2. Let $U \subset \mathbb{R}^d$ be a bounded open set and

$$\widehat{L}u = \sum_{|\alpha| \leq n} a_\alpha D^\alpha u$$

be a differential operator of order n with $a_\alpha \in C^\infty(\overline{U})$. Let $k \in \mathbb{Z}$, $1 \leq j \leq d$, $h \neq 0$, and $u \in H^{k+n}(U)$ with $\operatorname{supp}(u) \subset\subset U$, then

$$||(D_j^h \widehat{L} - \widehat{L} D_j^h)u||_{H^k(U)} \leq C_L \, ||u||_{H^{k+n}(U)}$$

for all $h \neq 0$ sufficiently small.

Proof. By the product rule of difference quotients,

$$D_j^h \widehat{L}u = \sum_{|\alpha| \leq n} D_j^h (a_\alpha D^\alpha u) = \underbrace{\sum_{|\alpha| \leq n} a_\alpha D^\alpha (D_j^h u)}_{\widehat{L} D_j^h u} + \underbrace{\sum_{|\alpha| \leq n} (D_j^h a_\alpha) \tau_{-he_j} (D^\alpha u)}_{(D_j^h \widehat{L} - \widehat{L} D_j^h)u}$$

Since each $a_\alpha \in C^\infty(\overline{U})$, there exists $C \geq 0$ and $C' \geq 0$ such that

$$||(D_j^h \widehat{L} - \widehat{L} D_j^h)u||_{H^k(U)} \leq C \sum_{|\alpha| \leq n} ||D^\alpha u||_{H^k(U)} \leq C' \, ||u||_{H^{k+n}(U)}$$

□

The two lemmas above combined are sufficient to establish regularity when $u \in H_0^1(\Omega)$ is supported in a coordinate neighbourhood.

Lemma 2.3 (Local Regularity, [3, Lemma 5.1.4])**.** Let $k \geq 0$, $u \in H^k(\Omega) \cap H_0^1(\Omega)$, and suppose that

1. $Lu \in H^{k-1}(\Omega)$.
2. There exists a chart (U, φ) such that $\operatorname{supp}(u) \subset\subset U$.
3. Either $U \cap \partial\Omega = \emptyset$, or $\varphi(U \cap \partial\Omega) \subset \{x_d = 0\}$.

then

$$||u||^2_{H^{k+1}(\Omega)} \leq C_{\Omega,L} (||Lu||^2_{H^{k-1}(\Omega)} + ||u||^2_{H^k(\Omega)}) \tag{2}$$

Proof. Let $\widehat{U} = \varphi(U)$ and $\widehat{u} = u \circ \varphi^{-1}$. Since u is supported in a coordinate patch, L has representation in local coordinates as

$$\widehat{Lu} = \widehat{L}\widehat{u} = -\sum_{m,n=1}^{d} g^{mn} D_m D_n \widehat{u} + \sum_{n=1}^{d} Y^n D_n \widehat{u} + \widehat{c}\widehat{u} \tag{3}$$

with each coefficient in $C^\infty(\overline{U})$. In addition, Equation 2 is equivalent to

$$||\widehat{u}||_{H^{k+1}(\widehat{U})} \leq C_{\Omega,L} (||\widehat{L}\widehat{u}||_{H^{k-1}(\widehat{U})} + ||\widehat{u}||_{H^k(\widehat{U})})$$

So all estimates can be computed in local coordinates. In particular, difference quotients become a viable method of estimating the $H^{k+1}(\widehat{U})$ norm of u.

If $k = 0$, then the lemma is shown by Lemma 2.1. Now suppose inductively that the lemma holds for k, $u \in H^{k+1}(\Omega)$, and $Lu \in H^k(\Omega)$. Since $\partial\Omega$ is given by $\{x_d = 0\}$ (if $U \cap \partial\Omega \neq \emptyset$), the difference quotient $D_j^h \widehat{u}$ is defined for each $1 \leq j \leq d-1$. By the inductive hypothesis applied to $D_j^h \widehat{u}$,

$$\begin{aligned} ||D_j^h \widehat{u}||_{H^{k+1}(\widehat{U})} &\leq C_{\Omega,L}(||\widehat{L}(D_j^h \widehat{u})||_{H^{k-1}(\widehat{U})} + ||\widehat{u}||_{H^{k+1}(\widehat{U})}) \\ &\leq C_{\Omega,L}(||D_j^h \widehat{L}\widehat{u}||_{H^{k-1}(\widehat{U})} + ||\widehat{u}||_{H^{k+1}(\widehat{U})}) \\ &+ C_{\Omega,L}||(D_j^h \widehat{L} - \widehat{L} D_j^h)\widehat{u}||_{H^{k-1}(\widehat{U})} \end{aligned}$$

By Lemma 2.2, the commutator can be estimated as

$$||(D_j^h \widehat{L} - \widehat{L} D_j^h)\widehat{u}||_{H^{k-1}(\widehat{U})} \leq C_L ||\widehat{u}||_{H^{k+1}(\widehat{U})}$$

So for all $h \neq 0$ sufficiently small,

$$||D_j^h \widehat{u}||^2_{H^{k+1}(\widehat{U})} \leq C_{\Omega,L}(||D_j^h \widehat{L}\widehat{u}||^2_{H^{k-1}(\widehat{U})} + ||\widehat{u}||^2_{H^{k+1}(\widehat{U})})$$

Sending $h \to 0$ yields

$$||D_j \widehat{u}||^2_{H^{k+1}(\widehat{U})} \leq C_{\Omega,L}(||D_j \widehat{L}\widehat{u}||^2_{H^{k-1}(\widehat{U})} + ||\widehat{u}||^2_{H^{k+1}(\widehat{U})})$$

so

$$||D_j \widehat{u}||^2_{H^{k+1}(\widehat{U})} \leq C_{\Omega,L}(||\widehat{L}\widehat{u}||^2_{H^{k}(\widehat{U})} + ||\widehat{u}||^2_{H^{k+1}(\widehat{U})}) \tag{4}$$

To deal with $||D_d u||_{H^{k+1}(\Omega)}$, it is sufficient to estimate $||D_j D_d u||_{H^k(\Omega)}$ for all $1 \leq j \leq d$. For each $1 \leq j \leq d-1$, since $D_j D_d u = D_d D_j u$, Equation 4 gives

$$||D_d D_j \widehat{u}||^2_{H^{k}(\widehat{U})} \leq ||D_j \widehat{u}||^2_{H^{k+1}(\widehat{U})} \leq C_{\Omega,L}(||\widehat{L}\widehat{u}||^2_{H^{k}(\widehat{U})} + ||\widehat{u}||^2_{H^{k+1}(\widehat{U})}) \tag{5}$$

Thus $D_d D_d u$ is the only remaining derivative to estimate. Since the matrix (g^{mn}) is positive definite, $g^{dd} > 0$. By compactness and Equation 3,

$$\begin{aligned} ||D_d D_d \widehat{u}||_{H^k(\widehat{U})} &\leq C_L ||\widehat{L}\widehat{u}||_{H^k(\widehat{U})} + C_L \sum_{m,n \neq (d,d)} ||D_m D_n \widehat{u}||_{H^k(\widehat{U})} \\ &+ C_L \sum_{m=1}^{d} ||D_m \widehat{u}||_{H^k(\widehat{U})} + C_L ||\widehat{u}||_{H^k(\widehat{U})} \\ &\leq C_{\Omega,L}(||\widehat{L}\widehat{u}||_{H^k(\widehat{U})} + ||\widehat{u}||_{H^{k+1}(\widehat{U})}) \end{aligned}$$

the derivative $D_d D_d u$ can be estimated in terms of every other second order term. Therefore Equation 4 and Equation 5 gives

$$||D_d \widehat{u}||^2_{H^{k+1}(\widehat{U})} \leq C_{\Omega,L}(||\widehat{L}\widehat{u}||^2_{H^{k}(\widehat{U})} + ||\widehat{u}||^2_{H^{k+1}(\widehat{U})}) \tag{6}$$

Finally, by combining Equation 4 and Equation 6,

$$||\widehat{u}||^2_{H^{k+2}(\widehat{U})} \leq C_{\Omega,L}(||\widehat{L}\widehat{u}||^2_{H^{k}(\widehat{U})} + ||\widehat{u}||^2_{H^{k+1}(\widehat{U})})$$

□

To transfer this local bound fully to Ω, it is sufficient to bound the commutator of L and multiplication by $\eta \in C^\infty(\Omega)$.

Lemma 2.4. Let $\eta \in C^\infty(\overline{\Omega})$ and $u \in H^k \cap H_0^1(\Omega)$, then

$$||L(\eta u) - \eta(Lu)||_{H^{k-1}(\Omega)} \le C_{L,\eta}(||u||_{H^k(\Omega)})$$

Proof.

$$\begin{aligned} L(\eta u) &= -\eta \Delta u - 2 \langle d\eta, du \rangle_g - u \Delta \eta + \eta(Xu) + u(X\eta) \\ &= \eta(Lu) - 2 \langle d\eta, du \rangle_g - u\Delta\eta + u(X\eta) \end{aligned}$$

so

$$||L(\eta u) - \eta(Lu)||_{H^{k-1}(\Omega)} \le C_{d\eta} ||u||_{H^k(\Omega)} + C_\eta ||u||_{H^{k-1}(\Omega)} \le C_\eta ||u||_{H^k(\Omega)}$$

□

Theorem 2.5 (Interior Regularity, [3, Proposition 5.1.9])**.** Let $k \ge 0$, $u \in H^k_{\text{loc}}(\Omega) \cap H^1_{\text{loc}}(\Omega)$, and suppose that $Lu \in H^{k-1}_{\text{loc}}(\Omega)$, then for any open sets $V \subset\subset U \subset\subset \Omega$,

$$||u||_{H^{k+1}(V)} \le C_{\Omega,L,U,V}(||Lu||_{H^{k-1}(U)} + ||u||_{H^k(U)})$$

Moreover, if $u \in H_0^1(\Omega)$, then the above also holds for $V \subset\subset U \subset\subset \overline{\Omega}$.

Proof. Let $\eta \in C_c^\infty(U)$, then $\eta u \in H_0^1(\Omega)$, and by Lemma 2.3 and Lemma 2.4,

$$\begin{aligned} ||\eta u||_{H^{k+1}(\Omega)} &\le C_{\Omega,L}(||L(\eta u)||_{H^{k-1}(\Omega)} + ||\eta u||_{H^k(\Omega)}) \\ &\le C_{\Omega,L}(||\eta(Lu)||_{H^{k-1}(\Omega)} + ||\eta(Lu) - L(\eta u)||_{H^{k-1}(\Omega)} + ||\eta u||_{H^k(\Omega)}) \\ &\le C_{\Omega,L,\eta}(||Lu||_{H^{k-1}(U)} + ||u||_{H^k(U)}) \end{aligned}$$

Let $\{(U_j, \varphi_j)\}_1^n$ be a family of charts covering V, and $\{\eta_j\}_1^n \subset C_c^\infty(U)$ be a partition of unity on V subordinate to $\{U_j\}_1^n$, then

$$||u||_{H^{k+1}(V)} \le \sum_{j=1}^n ||\eta_j u||_{H^{k+1}(\Omega)} \le C_{\Omega,L,U,V}(||Lu||_{H^{k-1}(U)} + ||u||_{H^k(U)})$$

□

If Ω is compact, then the local estimates translate directly into global estimates.

Theorem 2.6 (Elliptic Regularity Theorem, [3, Theorem 5.1.3])**.** Suppose that Ω is compact. Let $k \ge 0$, $u \in H^k(\Omega) \cap H_0^1(\Omega)$ with $Lu \in H^{k-1}(\Omega)$, then

$$||u||^2_{H^{k+1}(\Omega)} \le C_{\Omega,L}(||Lu||^2_{H^{k-1}(\Omega)} + ||u||^2_{H^k(\Omega)})$$

Corollary 2.7. For any $u \in H^1_{\text{loc}}(\Omega)$ such that $Lu \in C^\infty(\Omega)$, $u \in C^\infty(\Omega)$.

Corollary 2.8 ([3, Corollary 5.1.5])**.** The eigenfunctions of L belong to C^∞.

3 Existence of Solutions

To make use of the global regularity result, assume that Ω is compact in this section. For sufficiently large $\gamma \geq 0$, the energy estimate shows that

$$||u||^2_{H^1(\Omega)} \leq C_\Omega(\mathrm{Re}\,(\langle u, u\rangle_L) + \gamma\,||u||^2_{L^2(\Omega)}) \leq C_\Omega \mathrm{Re}(\langle u, u\rangle_{L+\gamma}) \tag{7}$$

so the sesquilinear form $\langle \cdot, \cdot\rangle_{L+\gamma}$ is coercive, so $(L+\gamma) : H^1_0(\Omega) \to H^{-1}(\Omega)$ is injective. Combining this with the identification $H^1_0(\Omega) \cong H^{-1}(\Omega)$ shows that $(L+\gamma)$ is in fact invertible.

Lemma 3.1 ([3, Proposition 5.1.1])**.** There exists $\gamma \geq 0$ such that

$$(L+\gamma) : H^1_0(\Omega) \to H^{-1}(\Omega) \quad u \mapsto \langle \cdot, u\rangle_{L+\gamma}$$

is an isomorphism.

Proof. By Equation 7, $||Lu||_{H^{-1}(\Omega)} \geq C_{\Omega,L}\,||u||_{H^1(\Omega)}$, so $(L+\gamma)$ coercive, has closed image, and admits a bounded inverse.

Suppose that $(L+\gamma)$ is not surjective, then there exists $0 \neq u_0 \in \mathrm{Im}(L+\gamma)^\perp$. By the Riesz representation theorem, there exists $0 \neq u^*_0 \in H^1_0(\Omega)$ such that $\langle u, u^*_0\rangle_{L+\gamma} = 0$ for all $u \in H^1_0(\Omega)$. In particular,

$$C_{\Omega,L}\,||u^*_0||_{H^1(\Omega)} \leq \mathrm{Re}(\langle u^*_0, u^*_0\rangle_{L+\gamma}) = 0$$

which is a contradiction. Therefore $(L+\gamma)$ is an isomorphism. □

If γ can be taken to be 0, then L is an isomorphism. In other cases, the invertibility of L can be inferred from the Fredholm alternative:

Theorem 3.2 (Existence of Weak Solutions, [3, Proposition 5.1.9])**.** The operator

$$L : H^1_0(\Omega) \to H^{-1}(\Omega)$$

is Fredholm of index zero. Thus it is an isomorphism if and only if L is injective.

Proof. Denote $T = L + \gamma$ and let $u \in H^1_0(\Omega)$, then $Tu = Lu + \gamma u$, so $u = T^{-1}Lu + \gamma T^{-1}u$ and $Lu = T(I - \gamma T^{-1})u$. In view of Theorem 2.6, T^{-1} is a bounded linear operator from $H^1_0(\Omega)$ to $H^3(\Omega) \cap H^1_0(\Omega)$. By Rellich's theorem, $\gamma T^{-1} : H^1_0(\Omega) \to H^1_0(\Omega)$ is compact, and L is a Fredholm operator of index zero. □

The existence theory can then be used to solve the homogeneous Dirichlet problem.

Proposition 3.3 ([3, Proposition 1.7])**.** Let $k \geq 0$ and suppose that $L : H^1_0(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, then there exists a unique bounded linear map

$$\mathrm{PI} : H^{k+1/2}(\partial\Omega) \to H^{k+1}(\Omega)$$

such that for any $g \in H^{k+1/2}(\partial\Omega)$, $L(\mathrm{PI}g) = 0$ and $\mathrm{PI}g|_{\partial\Omega} = g$.

Proof. Using an inverse of the trace operator $\tau^{-1} : H^{k+1/2}(\partial\Omega) \to H^{k+1}(\Omega)$, PI can be expressed as

$$\mathrm{PI}(g) = L^{-1}L\tau^{-1}g + \tau^{-1}g$$

where $L^{-1} : H^{k-1}(\Omega) \to H^{k+1}(\Omega)$ is bounded by Theorem 2.6. Given that L is an isomorphism, this inverse is uniquely determined. □

Finally, combining the existence theory in both cases allows solving the non-homogeneous Dirichlet problem.

Theorem 3.4 ([3, Equation 1.42])**.** Let $k \geq 0$ and suppose that $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism. For any $f \in H^{k-1}(\Omega)$ and $g \in H^{k+1/2}(\partial\Omega)$, there exists a unique $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ such that

$$\begin{cases} Lu = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

where

$$||u||_{H^{k+1}(\Omega)} \leq C_{L,\Omega}(||f||_{H^{k-1}(\Omega)} + ||g||_{H^{k+1/2}(\partial\Omega)} + ||u||_{H^k(\Omega)})$$

In particular, if $f \in C^\infty(\overline{\Omega})$ and $g \in C^\infty(\partial\Omega)$, then $u \in C^\infty(\overline{\Omega})$.

Proof. Let $u = L^{-1}f + \mathrm{PI}g$, then the estimate is given by Theorem 2.6 and Proposition 3.3. Since L is an isomorphism, this inverse is also uniquely determined. □

The Laplacian itself is a semipositive, self-adjoint operator. Combining this and the regularity result yields the spectral decomposition of $L^2(\Omega)$.

Proposition 3.5 ([3, Proposition 5.1.2])**.** Let $L = -\Delta$, then

1. There exists an orthonormal basis $\{u_j\}_1^\infty \subset L^2(\Omega) \cap C^\infty(\Omega)$ of eigenfunctions of L for $L^2(\Omega)$.
2. The eigenvalues $\{\lambda_j\}_1^\infty$ of L are non-negative and accumulate at $+\infty$.

Proof. For any $\phi, \psi \in H_0^1(\Omega)$ such that $L\phi, L\psi \in L^2(\Omega)$,

$$\langle L\phi, \psi\rangle_{L^2(\Omega)} = \langle d\phi, d\psi\rangle_{L^2(\Omega;T\Omega)} = \langle \phi, L\psi\rangle_{L^2(\Omega)}$$

By Lemma 3.1, $T = L + \gamma$ is an isomorphism for sufficiently large $\gamma \geq 0$. For any $u, v \in L^2(\Omega)$, there exists $\phi, \psi \in H_0^1(\Omega)$ with $u = T\phi$ and $v = T\psi$.

$$\begin{aligned} \langle T^{-1}u, v\rangle_{L^2(\Omega)} &= \langle T^{-1}T\phi, T\psi\rangle_{L^2(\Omega)} = \langle \phi, T\psi\rangle_{L^2(\Omega)} \\ &= \langle T\phi, \psi\rangle_{L^2(\Omega)} = \langle u, T^{-1}v\rangle_{L^2(\Omega)} \end{aligned}$$

Since the embedding $H^2(\Omega) \to L^2(\Omega)$ is compact,

$$T^{-1}|_{L^2(\Omega)} : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega)$$

is a compact self-adjoint operator. Thus there exists an orthonormal basis $\{u_j\}_1^\infty \subset L^2(\Omega)$ of eigenfunctions of T^{-1} for $L^2(\Omega)$, and the eigenvalues $\{\mu_j\}_1^\infty$ of T^{-1} accumulate at 0. For each $j \in \mathbb{N}$, $T^{-1}u_j = \mu_j u_j$ implies that

$$Lu_j = \lambda_j u_j \quad \text{where} \quad \lambda_j = \frac{1}{\mu_j} - \gamma$$

By Theorem 2.6, $\{u_j\}_1^\infty \subset L^2(\Omega) \cap C^\infty(\Omega)$. In addition,

$$\begin{aligned}\langle Tu_j, u_j\rangle_{L^2(\Omega)} &= \langle du_j, du_j\rangle_{L^2(\Omega;T\Omega)} + \gamma\,||u||^2_{L^2(\Omega)} \\ &\geq \langle du_j, du_j\rangle_{L^2(\Omega;T\Omega)} = \langle Lu_j, u_j\rangle_{L^2(\Omega)} \geq 0\end{aligned}$$

so each $u_j, \lambda_j \geq 0$. Given that $\{u_j\}_1^\infty$ accumulates at 0, $\{\lambda_j\}_1^\infty$ accumulates at $+\infty$. ☐

The sesquilinear form allows expressing the eigenvalues as minimums over subspaces of $H_0^1(\Omega)$, and the eigenfunctions as the minimisers. In addition, the smallest eigenvalue of L corresponds to a sharp estimate of the constant for Pointcaré's inequality.

Theorem 3.6 (Max-Min Theorem, [1, Section 1.5], [3, Exercise 5.1.2])**.** Let $L = -\Delta$, $\{\lambda_j\}_1^\infty$ be an increasing enumeration of its spectrum, repeated based on multiplicity, and $\{u_j\}_1^\infty \subset L^2(\Omega)$ be the corresponding eigenfunctions, then for each $m \in \mathbb{N}$,

$$\lambda_m = \min_{\substack{u \in H_0^1(\Omega)\setminus\{0\} \\ u \perp \{u_j\}_1^{m-1}}} \frac{\langle u, u\rangle_L}{||u||^2_{L^2(\Omega)}}$$

where the minimum is achieved when u is an eigenfunction for λ_m. Moreover,

$$\lambda_m = \max_{\substack{N \subset H_0^1(\Omega) \\ \dim N < m}} \min_{\substack{u \perp N \\ u \neq 0}} \frac{\langle u, u\rangle_L}{||u||^2_{L^2(\Omega)}}$$

where the maximum is achieved when $N = \operatorname{span}(u_j : 1 \leq j \leq m-1)$. In particular,

$$\lambda_1 = \min_{u \in H_0^1(\Omega)} \frac{\langle u, u\rangle_L}{||u||^2_{L^2(\Omega)}} = \min_{u \in H_0^1(\Omega)} \frac{\langle du, du\rangle_{L^2(\Omega;T\Omega)}}{||u||^2_{L^2(\Omega)}}$$

If every connected component of Ω has non-empty boundary, then $\lambda_1 > 0$, and for any $u \in H_0^1(\Omega)$,

$$||u||^2_{L^2(\Omega)} \leq \frac{1}{\lambda_1}\,||du||^2_{L^2(\Omega;T\Omega)}$$

with equality when u is an eigenfunction for λ_1.

Proof. For any $u \in H_0^1(\Omega)$, write $u = \sum_{j\in\mathbb{N}} a_j u_j$. For each $n \geq m$, applying the bilinear form to $(u - \sum_{j=1}^n a_j u_j)$ shows that

$$\begin{aligned}0 &\leq \langle u, u\rangle_L - 2\sum_{j=1}^n a_j\lambda_j \langle u, u_j\rangle_{L^2(\Omega)} + \sum_{j,k=1}^n a_j a_k \lambda_j \langle u_j, u_k\rangle_{L^2(\Omega)} \\ &= \langle u, u\rangle_L - \sum_{j=m}^n a_j\lambda_j \langle u, u_j\rangle_{L^2(\Omega)}\end{aligned}$$

so

$$\lambda_m \|u\|^2_{L^2(\Omega)} \leq \sum_{j=m}^\infty \lambda_j a_j^2 \leq \langle u, u\rangle_L = \langle du, du\rangle_{L^2(\Omega;T\Omega)}$$

where equality holds when u is an eigenfunction corresponding to λ_m.
Now, let $N \subset H_0^1(\Omega)$ be a $(m-1)$-dimensional subspace, and
$M = \mathrm{span}(u_j : 1 \leq j \leq m)$, then there exists $v \in M$ with $v \perp N$. In this case,

$$\min_{\substack{u\perp N\\ u\neq 0}} \frac{\langle du, du\rangle_{L^2(\Omega;T\Omega)}}{\|u\|^2_{L^2(\Omega)}} \leq \frac{\langle dv, dv\rangle_{L^2(\Omega;T\Omega)}}{\|u\|^2_{L^2(\Omega)}} \leq \lambda_m$$

If every connected component of Ω has non-empty boundary, then
$\langle Lu, u\rangle_{L^2(\Omega)} > 0$ for all $u \in H_0^1(\Omega)$ with $u \neq 0$, so $\lambda_1 > 0$. □

A Results Used

Lemma A.1 (Cauchy's Inequality). Let $a, b \geq 0$ and $\varepsilon > 0$, then

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$$

Theorem A.2 (Pointcaré's Inequality, [3, Proposition 4.5.2]). If every connected component of Ω has non-empty boundary, then for any $u \in H_0^1(\Omega)$,

$$||u||_{L^2(\Omega)} \leq C_\Omega \, ||du||_{L^2(\Omega)}$$

Theorem A.3 (Rellich's Theorem, [3, Proposition 4.3.4]). If Ω is compact, then for any $s \in \mathbb{R}$ and $\sigma > 0$, the inclusion

$$\iota : H^{s+\sigma}(\Omega) \to H^s(\Omega)$$

is compact.

Theorem A.4 (Green's Formulas). Let $u, v \in C^\infty(\overline{\Omega})$, then

$$\langle u, -\Delta v \rangle_{L^p(\Omega)} = \langle du, dv \rangle_{L^p(\Omega; T\Omega)} - \langle u, \partial_\nu v \rangle_{L^p(\partial\Omega)}$$

and

$$\int_\Omega u\Delta v - v\Delta u dV = \int_{\partial\Omega} v\partial_\nu u - u\partial_\nu v dS$$

In particular, if $u, v \in C_c^\infty(\Omega)$, then

$$\langle u, -\Delta v \rangle_{L^p(\Omega)} = \langle du, dv \rangle_{L^p(\Omega; T\Omega)}$$

Proposition A.5 (Inverse of Trace, [3, Proposition 4.1.7]). Let $s > 0$, then there exists a bounded linear operator

$$\tau^{-1} : H^s(\partial\Omega) \to H^{s+1/2}(\Omega)$$

such that for any $u \in C^\infty(\partial\Omega)$, $\tau^{-1}u|_{\partial\Omega} = u$.

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